Averaged Reynolds Equation for Flows between Rough Surfaces in Sliding Motion

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Abstract. The flow between rough surfaces in sliding motion with contacts between these surfaces, is analyzed through the volume averaging method. Assuming a Reynolds (lubrication) approximation at the roughness scale, an average flow model is obtained combining spatial and time average. Time average, which is often omitted in previous works, is specially discussed. It is shown that the effective transport coefficients, traditionally termed 'flow factors' in the lubrication literature, that appear in the average equations can be obtained from the solution to two closure problems. This allows for the numerical determination of flow factors on firmer bases and sheds light on some arguments to the literature. Moreover, fluid flows through fractures form an important subset of problems embodied in the present analysis, for which macroscopisation is given.

Key words: volume averaging, Reynolds equations, lubrication, rough surfaces, fracture flows.

Nomenclature

a	local distance between surfaces.
A_{sf}	is the boundary of the contact zones within S_f.
$\mathcal{A}_{sf}$	is the vertical projection of A_{sf} in the Euclidean plane Oxy.
$\mathbf{b}$, $\mathbf{c}_i$, $\mathbf{c}$	closure vectors fields for the pressure with $i = 1, 2$.
$\mathbf{C}$	macroscopic transport tensor.
h_i	height of solid surface number $i = 1, 2$.
h_{0i}	mean height of solid surface number $i = 1, 2$.
h_m	$h_{02} - h_{01}-$ mean distance between surfaces.
h	local aperture between surfaces.
K	local permeability field.
$\mathbf{K}^*$	macroscopic permeability tensor.
$\mathbf{n}_{sf}$	normal vector pointing out of a subscript sf domain.
Ω	is the mean surface defined by $h_+(\mathbf{x}) = (h_1 + h_2)/2$.
Ω_f	is the region of Ω where the local aperture is non zero.
Ω_{sf}	is the boundary of the contact zones within Ω.
p	pressure field.
ϕ_x, ϕ_{fp}	are the Poiseuille flow factor.
$\phi_s, \phi_f, \phi_{fs}$	are the Couette flow factor.
$\mathbf{q}$	the volumetric flow rate per unit width.
S	is an elementary representative region of Ω.
S_f	is the region of S where the local aperture is non zero.

σ_i	is the root mean square roughness of surface $i = 1, 2$.
σ	$= \sqrt{\sigma_1^2 + \sigma_2^2}$ is the composite roughness.
τ	shear vector which is the shear stress tensor projected parallel to the surfaces mean planes.

1. Introduction

1.1. CONTEXT AND AIM OF THE STUDY

Studying the effect of surface roughness on lubrication is a very complex tribological problem. Firstly it involves a time dependent fluid domain resulting from the motion and deformation of the moving solid surfaces. This geometrical complexity is in itself more or less difficult to analyze, depending on the surface roughness patterns. Additional physical effects, such as cavitation, piezoviscosity or compressibility, among others, may contribute to complexity of the problem. Such effects could moreover be investigated at the various scales for which they occur, that is for a few asperities, for the statistically representative region, or for the system scale. In this paper we have nevertheless concentrated on the geometrical complexity of the problem, in the aim of finding an average influence of the surface roughness. Noting that the typical roughness length is much smaller than the surface size, it is not surprising that many authors have been interested by an average flow description. This can be done by using an up-scaling procedure which derives macroscopic equations which are then used to predict the average flow. It is well known (Whitaker, 1999) that such up-scaling depends on the relevant microscopic equations.

Here, the micro-scale is the surface roughness scale. In most applications the roughness local slope is small. It is therefore usually assumed that the flow equations at the roughness scale are well described by the Reynolds (lubrication) approximation. We will use this approximation in this study and will discuss it in more detail in Section 1.2. Deriving the average flow model under this assumption can be done by using various techniques.

In the context of surface lubrication, the first developments were made by Christensen (1970) and Chow and Cheng (1976) within the framework of the stochastic process theory. These works were limited to two-dimensional transverse and longitudinal roughness. Patir and Cheng (1978, 1979), were the first to propose a model for general roughness patterns. Their derivation was essentially heuristic, and it fails to properly model the situations where the roughness anisotropy directions are not identical to the Cartesian axis. This was pointed out for the first time by Elrod (1979) and subsequently by various authors, including Tripp (1983) who derived the correct tensorial form of the average flow model using a stochastic approach. When the off-diagonal terms of the tensor are negligible, Tripp model is, however, essentially identical to that of Patir and Cheng. It may be noted that Tripp did not consider the possibility of contact between surfaces. Under the no-contact assumption, the lubrication problem was addressed by Bayada and Chambat (1988) and Bayada and Faure (1989) within the framework of the homogenization theory for spatially periodic structures. The Patir and Cheng model was again recovered when the off-diagonal terms of tensors could be neglected. In addition to establishing the average flow model on a much firmer base, one interesting feature of homogenization is to propose local problems, called auxiliary problems, that have to be solved over a unit cell of the periodic micro-structure in order to compute the average tensors (see for instance Mei and Auriault, 1989).

Defining such auxiliary problems is a key technical step, very similar to finding closure problems for the volume averaging technique used in the present paper. Moreover very similar technical steps and conclusions can also be found from an other work using a very similar volume averaging technique in periodic flows, in the context of suspension shear flow (Adler and Brenner, 1985; Adler *et al.*, 1985).

However, in contrast to assumptions made in Bayada *et al.* or Tripp (1983), the present work considers cases where solid contact areas (between lubricated surfaces) do not involve any fluid. There is also an additional feature that distinguishes the present work from the previous ones. We consider situations where the sliding motion of surfaces may lead to combining time and spatial averaging in order to obtain the macroscopic behavior. An example of such a situation is considered in a companion paper (Letalleur *et al.*, 2000).

Similarly with mostly all previous literature, additional complications due to contact, surface deformation (see, however, Knoll *et al.*, 1998) or physical effects previously listed (cavitation, piezoviscosity,..) are ignored. The fluid is then assumed incompressible, isothermal and the viscosity is constant.

Although the present work was motivated by lubrication oriented problems, it is also of interest for the modeling of flow in a single fracture induced by average pressure gradients, see Adler and Thovert (1999).

This case is also to be observed when the velocities of the surfaces are set to zero. It also presents some formal analogy (again when the surfaces are assumed to be motionless) with the modeling of single phase flow in heterogeneous porous media as in Quintard and Whitaker (1987). The method we used, which was initially developed by Whitaker and co-workers, has been extensively used for studying transport phenomena in porous media, see Whitaker (1999) and references therein. It is of interest to apply it to a somewhat new domain (such as lubrication). The possible time-dependence of the spatial averaged variables is an interesting feature of the present application. Although time-dependence can be rather simply dealt with in our context, this is an example of situations involving space and time averages (two phase flows in porous media or rough fractures are obvious examples of such situations).

In this respect, our approach presents striking similarities with works done in the area of suspensions (Adler and Brenner 1985; Adler *et al.* 1985). In particular, there are considerations on time averaging and ergodicity which may be translated in the present case, and will be discussed in Section 3.

Finally, the paper is organized as follows. The main assumptions are listed in Section 1.2 where the equations governing the flow at the roughness scale are also presented. The volume averaging is performed in Section 2 so as to deduce the closure problems derived in Section 2.1. The average flow model is obtained in Section 2.2 and compared with the classical Patir and Cheng results. Section 3 is devoted to the discussion results. The expression of flow factors turns out to be a particular case (see Section 4, the case of simple unidirectional striated surfaces) which can be derived from our model.

1.2. PROBLEM FORMULATION AND HYPOTHESIS

The situation under study is sketched in Figure 1. This paper considers random rough surfaces, for which the scale of the 'macroscopic' geometry and the 'microscopic' roughness are greatly distinct both being spatially variable. The physical situation considered in this paper is when the two surfaces are sliding with different parallel velocities and simultaneously an overall pressure drop is applied on the fluid. Hence the aim of this study is to obtain some macroscopic description for hydrodynamical quantities at the macroscopic level, for which the spatial variations of the aperture field are very slow compared to the microscopic ones. It is assumed that the mean planes of the two surfaces are parallel. The velocity of the top surface (surface 2) is $\mathbf{U}_2$. That of the bottom surface (surface 1) is $\mathbf{U}_1$. Note that $\mathbf{U}_2$ and $\mathbf{U}_1$ are not necessarily collinear. A reference plane $z = 0$ is introduced and each surface is described by

$$h_i = h_{0i} + \widetilde{h}_i, \qquad i = 1, 2,$$

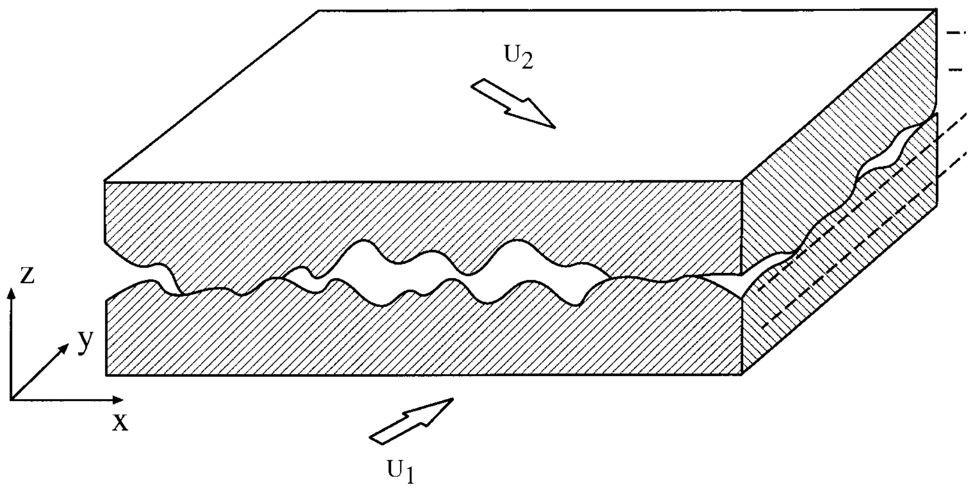

Figure 1. System of two rough surfaces in sliding motion.

where h_{0i} is the mean plane of surface i. $\widetilde{h}_i$ is the height variation of surface i around its mean plane. The mean distance between the two surfaces is then given by

$$h_m = h_{02} - h_{01},$$

and the local distance between both surfaces by

$$a = h_{02} + \widetilde{h}_2 - (h_{01} + \widetilde{h}_1).$$

When h_m decreases, the two surfaces are in contact. Usually, in order to determine the new shapes of the two surfaces one has to take their deformation into account. This phenomenon is, however, not considered in the present paper. In accordance with previous works (see the work of Adler and Thovert (1999) and the references therein), the local aperture field h is simply defined by

$$h = a, \quad \text{if} \quad a \geqslant 0,$$

$$h = 0, \quad \text{if} \quad a < 0.$$

The zones where $a < 0$ are the contact zones. The fluid is assumed to be Newtonian and incompressible. The viscosity is constant. Temperature variations and cavitation phenomena, if any, are not considered. As mentioned before, the local slopes are assumed to be small and the flow at the roughness scale is then governed by the Reynolds (lubrication) equation. This can be rigorously established when there is no contact between the two solid surfaces (Adler and Thovert, 1999). The relevance of the Reynolds equation, when surfaces moving at different speeds touch one another, is not obvious. At contact points, a candid geometrical point of view should attribute two possible velocities to the fluid namely U_1 and U_2 ! On the other hand, it seems very sensible to admit the validity of Reynolds equation up to some phenomenological distance to the solid contacts. Such distance might be small, and its exact value should hardly influence the macroscopic behavior, except for those related to the shear stress undergone by surfaces. Nevertheless, to the best of our knowledge, the literature does not provide any estimate for such continuum mechanics cut-off. Consequently, we postulate the validity of the Reynolds equation as it has almost always been done in previous works (Patir and Cheng, 1979; Peeken *et al.*, 1997).

Owing to the relative motion of surface, the problem is unsteady. Using the Reynolds equation implies that the characteristic time t_d of momentum diffusion over a distance of the order of roughness heights is small in comparison to the characteristic time t_s associated with the relative motion of surfaces. These times can be estimated respectively as,

$$t_\mathrm{d} = \frac{\sigma^2}{\nu}, \qquad t_\mathrm{s} = \frac{l_\mathrm{c}}{(U_2 - U_1)},$$

where σ is a roughness scale (for instance the composite standard deviation of $\widetilde{h}_1$ and $\widetilde{h}_2$, see Equation (36)) and ν is the kinematic viscosity of the fluid. l_c may be regarded as the correlation length of the aperture field. This yields the following result

$$\frac{t_\mathrm{d}}{t_\mathrm{s}} = \frac{\sigma\,(U_2 - U_1)}{\nu}\frac{\sigma}{l_\mathrm{c}},$$

which shows that a sufficient condition for the quasi-steadiness approximation to be valid is related to a regime of small reduced Reynolds number that is $\mathrm{R}e = \sigma^2(U_2 - U_1)/\nu l_\mathrm{c} \ll 1$. This constraint is not difficult to satisfy since σ is usually very small (of the order of 10 μm or less).

Under the previously mentioned assumptions, the governing equations and boundary conditions at the scale roughness of the roughness height are given by the classical lubrication approximation

$$\begin{aligned} &\mathbf{q} = -\tfrac{h^3}{12\mu}\nabla p + (\mathbf{U}_2 + \mathbf{U}_1)\,\tfrac{h}{2}, \quad \text{in} \quad \Omega_\mathrm{f} \\ &\tfrac{\partial h}{\partial t} + \nabla\cdot q = 0, \quad \text{in} \quad \Omega_\mathrm{f} \\ &\mathbf{q}\cdot\mathbf{n}_\mathrm{sf} = 0, \quad \text{at} \quad \Omega_\mathrm{sf} \end{aligned} \tag{1}$$

in which $\mathbf{q}$ is the volumetric flow rate per unit width, Ω is the mean surface defined by $h_+(\mathbf{x}) = (h_1 + h_2)/2$, Ω_f denotes the region of Ω occupied by the fluid, (i.e., the region where $h > 0$), Ω_sf denotes the boundary of the contact zones within Ω, $\partial\Omega_\mathrm{f}$ denotes the region of the boundary of Ω where $h > 0$ and $\mathbf{n}_\mathrm{sf}$ represents the unit normal vector pointing from the solid-phase toward the fluid-phase at Ω_sf.

Because the deformation of surfaces is ignored, there is a simple relation between the time and space derivatives of h_i,

$$\frac{Dh_i}{Dt} = \frac{\partial h_i}{\partial t} + \mathbf{U}_i\cdot\nabla h_i = 0,$$

with $i = 1, 2$ which give

$$\frac{\partial h}{\partial t} = \frac{\partial h_2}{\partial t} - \frac{\partial h_1}{\partial t} = -\mathbf{U}_2\cdot\nabla h_2 + \mathbf{U}_1\cdot\nabla h_1.$$

This leads to the simplified roughness scale equations

$$\mathbf{q} = -\frac{K}{12\mu}\nabla p + \frac{(\mathbf{U}_1 + \mathbf{U}_2)}{2}h, \quad \text{in} \quad \Omega_\mathrm{f} \tag{2}$$

$$\nabla\cdot\mathbf{q} = \mathbf{U}_2\cdot\nabla h_2 - \mathbf{U}_1\cdot\nabla h_1, \quad \text{in} \quad \Omega_\mathrm{f} \tag{3}$$

$$\mathbf{q}\cdot\mathbf{n}_\mathrm{sf} = 0, \quad \text{at} \quad \Omega_\mathrm{sf} \tag{4}$$

in which $K = h^3$.

2. Volume Averaging

The method of volume averaging (see Whitaker, 1999 and references therein) begins by associating with every point in space (in both the fluid-phase and the solid-phase) an averaging volume (although this is an averaging surface in the case we are studying, we will call it an averaging 'volume') denoted by S (so as to recall that we are dealing with surfaces). The surface S is part of the total domain Ω. Such a 'volume' is illustrated in Figure 2 where we have located the centroid of the averaging volume by the position vector $\mathbf{x}$, the radius of the averaging volume by r_0. Two averages are used in this method. The first is the superficial volume average which can be expressed as

$$\langle \psi_{\mathrm{f}} \rangle = \frac{1}{S} \int_{S_{\mathrm{f}}} \psi_{\mathrm{f}} \,\mathrm{d}S,$$

in which ψ_{f} is any function associated with the fluid-phase. S_{f} is the surface of the fluid-phase contained within the averaging surface S. The second is the intrinsic volume average which is defined by

$$\langle \psi_{\mathrm{f}} \rangle^{f} = \frac{1}{S_{\mathrm{f}}} \int_{S_{\mathrm{f}}} \psi_{\mathrm{f}} \,\mathrm{d}S.$$

A basic tool of the volume averaging method is the spatial averaging theorem Howes and Whitaker (1985) which allows one to interchange differentiation and integration on Euclidean geometrical domains

$$\langle \nabla \psi_{\mathrm{f}} \rangle = \nabla \langle \psi_{\mathrm{f}} \rangle + \frac{1}{S} \int_{\mathcal{A}_{\mathrm{sf}}} \psi_{\mathrm{f}} \mathbf{n}_{\mathrm{sf}} \,\mathrm{d}\mathcal{A}_{\mathrm{sf}}, \tag{5}$$

where $\mathcal{A}_{\mathrm{sf}}$ denotes the boundary of the contact zones within the Euclidean plane associated with Cartesian Oxy coordinates and S the corresponding surface in

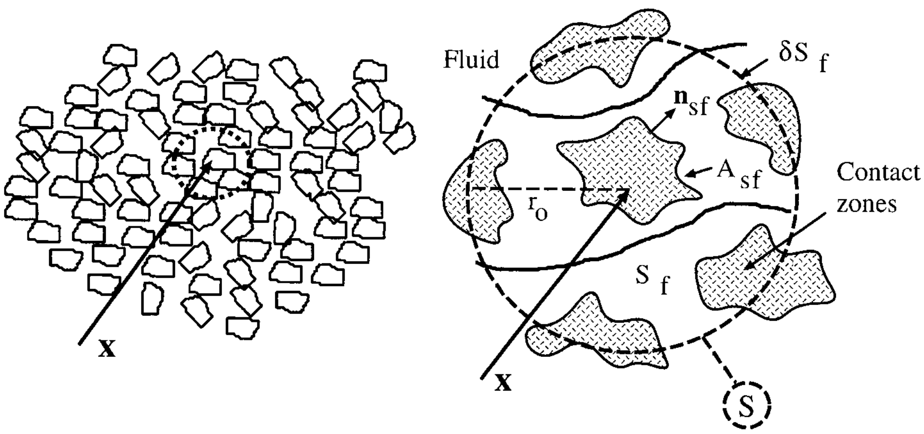

Figure 2. Averaging 'volume'.

the Euclidean plane. This result can be generalysed on non-Euclidean geometrical domains (Gray, 1993) such as Ω and Ω_{fs}. Nevertheless, such complications are unnecessary in the context we are dealing with, since in the case of a small slope it can be shown, that differentiation and integration on Ω and Ω_{f} can be approximated self-consistently using Euclidean rules. More precisely, Appendix A shows how this can be rigorously deduced from a small slope ϵ expansion, with the same approximation as the one used to obtain Reynolds equations. For now, as was previously done when discarding $O(\epsilon^2)$ terms in using the Reynolds approximation, we will use the standard volume average integro-differential machinery, without explicit mentioning of its $O(\epsilon^2)$ character. Hence, the Equation (5) can be rewritten as

$$\langle \nabla \psi_{\text{f}} \rangle = \nabla \langle \psi_{\text{f}} \rangle + \frac{1}{S} \int_{A_{\text{sf}}} \psi_{\text{f}} \mathbf{n}_{\text{sf}} \, \mathrm{d}A, \tag{6}$$

Where, A_{sf} now denotes the boundary of the contact zones within Ω_{f} of Ω. Hence, the vectorial version of (6) on S_{f} is simply

$$\langle \nabla \cdot \boldsymbol{\psi}_{\text{f}} \rangle = \nabla \cdot \langle \boldsymbol{\psi}_{\mathbf{f}} \rangle + \frac{1}{S} \int_{A_{\text{sf}}} \boldsymbol{\psi}_{\mathbf{f}} \cdot \mathbf{n}_{\text{sf}} \, \mathrm{d}A.$$

Volume averaging begins by forming the superficial average of Equation (3)

$$\langle \nabla \cdot \mathbf{q} \rangle = \langle \mathbf{U}_2 \cdot \nabla h_2 - \mathbf{U}_1 \cdot \nabla h_1 \rangle,$$

which leads to

$$\langle \nabla \cdot \mathbf{q} \rangle = \mathbf{U}_2 \cdot \langle \nabla h_2 \rangle - \mathbf{U}_1 \cdot \langle \nabla h_1 \rangle. \tag{7}$$

Interchanging differentiation and integration in Equation (7) is accomplished by means of the spatial averaging theorem. Using this result and taking into account the boundary condition (4) lead to

$$\nabla \cdot \langle \mathbf{q} \rangle = \mathbf{U}_2 \cdot \langle \nabla h_2 \rangle - \mathbf{U}_1 \cdot \langle \nabla h_1 \rangle. \tag{8}$$

At this point, it is worth noting that K, h, ∇h_2, ∇h_1 in Equation (3), depend not only on the space coordinates but also on time. However, these equations are free of time derivatives. Although here S_{f} is time dependent (in the case where contact zones are to be found), this does not introduce any particular difficulty in the spatial averaging process (the problem is in fact quasi-steady). Averaging Equation (2–4) leads to

$$\langle \mathbf{q} \rangle = -\frac{1}{12\mu} \langle K \nabla p \rangle + \frac{\mathbf{U}_2 + \mathbf{U}_1}{2} \langle h \rangle. \tag{9}$$

At this point, it is useful to introduce the following decomposition (Gray, 1975) for the pressure, $p = \langle p \rangle^f + \widetilde{p}$, in which $\widetilde{p}$ is referred to as the spatial deviation pressure. Using this decomposition, into the term $\langle K \nabla p \rangle$ from Equation (9) enables us to write

$$\langle K \nabla p \rangle = \left\langle K \nabla \langle p \rangle^f \right\rangle + \langle K \nabla \widetilde{p} \rangle.$$

It becomes, following (Whitaker, 1999):

$$\langle K\nabla p\rangle = \langle K\rangle\nabla\langle p\rangle^f + \langle K\nabla\widetilde{p}\rangle \tag{10}$$

when constraint $r_0/L_p \ll 1$ is satisfied with L_p representing the characteristic length associated with the average pressure. Substituting Equation (9) for (10) leads to

$$\langle \mathbf{q}\rangle = -\frac{1}{12\mu}(\langle K\rangle\nabla\langle p\rangle^f + \langle K\nabla\widetilde{p}\rangle) + \frac{\mathbf{U}_2+\mathbf{U}_1}{2}\langle h\rangle \tag{11}$$

2.1. CLOSURE PROBLEMS

In order to obtain a closed form of Equation (11), we develop a representation for the spatial pressure deviation $\widetilde{p}$. To this end we subtract Reynolds Equation (2) from its average (11)

$$\begin{aligned}\widetilde{\mathbf{q}} &= \mathbf{q} - \langle\mathbf{q}\rangle \\ &= -\frac{1}{12\mu}(\widetilde{K}\nabla\langle p\rangle^f + K\nabla\widetilde{p} - \langle K\nabla\widetilde{p}\rangle) + \frac{\mathbf{U}_2+\mathbf{U}_1}{2}\widetilde{h}, \quad \text{in } \Omega_\mathrm{f}\end{aligned} \tag{12}$$

in which $\widetilde{\mathbf{q}}$ represents spatial deviation of the volumetric flow rate per unit width, $\widetilde{K} = K - \langle K\rangle$, $\widetilde{h} = h - \langle h\rangle$. Similarly, for the mass conservation from (3) and (8)

$$\begin{aligned}\nabla\cdot\widetilde{\mathbf{q}} &= \nabla\cdot\mathbf{q} - \nabla\cdot\langle\mathbf{q}\rangle \\ &= \mathbf{U}_2\cdot(\nabla h_2 - \langle\nabla h_2\rangle) - \mathbf{U}_1\cdot(\nabla h_1 - \langle\nabla h_1\rangle) \quad \text{in} \quad \Omega_f\end{aligned} \tag{13}$$

It may be observed that the boundary condition given by Equation (4) is in fact necessarily verified since $h = 0$ at Ω_sf. Therefore, at this stage, it is not necessary to deduce a boundary condition for the spatial deviation.

The next step consists in considering that it is sufficient to determine the spatial deviations $\widetilde{p}$ and $\widetilde{\mathbf{q}}$ over a local representative region of the heterogeneous system. This classically leads to treat the representative region as a unit cell in a spatially periodic system (Whitaker, 1999) and therefore to impose the following conditions

$$\widetilde{p}(\mathbf{r}+l_i) = \widetilde{p}(\mathbf{r}), \qquad \widetilde{\mathbf{q}}(\mathbf{r}+l_i) = \widetilde{\mathbf{q}}(\mathbf{r}), \qquad i = x, y$$

in which l_i represents the two non-unique lattice vectors required to describe a spatially periodic system in two-dimensions. $\mathbf{r}$ is the vector of the position enabling to locate any point in the fluid-phase. For the spatial deviations $\widetilde{p}$ and $\widetilde{\mathbf{q}}$ to be locally periodic from a spatial point of view, it is also necessary that the source terms $\mathbf{U}_1$, $\mathbf{U}_2$ and $\nabla\langle p\rangle^f$ in Equations (12) and (13) should be considered as constants over the representative region. This is clear for $\mathbf{U}_1$, $\mathbf{U}_2$ since the surface velocities are assumed to be constant in the whole system. Appendix B shows that $\nabla\langle p\rangle^f$ can also be regarded as a constant, provided that $r_0/L_p \ll 1$. When this constraint is satisfied, the closure problem can therefore be expressed as

$$\begin{aligned}
\widetilde{\mathbf{q}} &= -\frac{1}{12\mu}(\widetilde{K}\nabla\langle p\rangle^f + K\nabla\widetilde{p} - \langle K\nabla\widetilde{p}\rangle) + \frac{\mathbf{U}_2+\mathbf{U}_1}{2}\widetilde{h}, \quad \text{in} \quad S_\mathrm{f} \\
\nabla\cdot\widetilde{\mathbf{q}} &= \nabla\cdot\mathbf{q} - \nabla\cdot\langle\mathbf{q}\rangle \\
&= \mathbf{U}_2\cdot(\nabla h_2 - \langle\nabla h_2\rangle) - \mathbf{U}_1\cdot(\nabla h_1 - \langle\nabla h_1\rangle), \quad \text{in} \quad S_\mathrm{f}
\end{aligned} \tag{14}$$

or

$$\begin{aligned}
-\frac{1}{12\mu}(\nabla\cdot(K\nabla\widetilde{p}) \ - \ \nabla\cdot\langle K\nabla\widetilde{p}\rangle) &= \frac{1}{12\mu}\nabla\cdot(\widetilde{K}\nabla\langle p\rangle^f) + \frac{\mathbf{U}_2+\mathbf{U}_1}{2}\cdot\nabla\widetilde{h} \\
&+ \ \mathbf{U}_2(\nabla h_2 - \langle\nabla h_2\rangle) - \mathbf{U}_1\cdot(\nabla h_1 - \langle\nabla h_1\rangle), \quad \text{in} \quad S_\mathrm{f}
\end{aligned} \tag{15}$$

In addition, as is usually done in the volume averaging method (Whitaker, 1999), we assume that $\langle\widetilde{p}\rangle^f = 0$. The form of the boundary value problem for $\widetilde{p}$ suggests a representation for $\widetilde{p}$ given by

$$\widetilde{p} = \mathbf{b}\cdot\nabla\langle p\rangle^f + \mu\,\mathbf{c}_2\cdot\mathbf{U}_2 + \mu\,\mathbf{c}_1\cdot\mathbf{U}_1 + \varphi \tag{16}$$

where φ is an arbitrary function. $\mathbf{b}$ $\mathbf{c}_1$, $\mathbf{c}_2$ are the closure variables. As φ is an arbitrary function, we are free to specify $\mathbf{b}$, $\mathbf{c}_1$, $\mathbf{c}_2$ by means of the following three boundary value problems that are suggested by substituting the closure Equation (16) into the pressure deviation Equation (15)

Problem 1

$$\begin{aligned}
&\nabla\cdot(\widetilde{K}\mathbf{I} + K\nabla\mathbf{b} - \langle K\nabla\mathbf{b}\rangle) = 0, \quad \text{in} \quad S_\mathrm{f}, \\
&\mathbf{b}(\mathbf{r}+l_i) = \mathbf{b}(\mathbf{r}), \quad i = x, y, \\
&\langle\mathbf{b}\rangle^f = 0.
\end{aligned} \tag{17}$$

Problem 2 and 3

$$\begin{aligned}
&\tfrac{1}{12}\nabla\cdot(K\nabla\mathbf{c}_j - \langle K\nabla\mathbf{c}_j\rangle) = \theta_j(\langle\nabla h_j\rangle - \nabla h_j) + \tfrac{1}{2}\nabla\widetilde{h}, \quad \text{in} \quad S_\mathrm{f} \\
&\mathbf{c}_j(\mathbf{r}+l_i) = \mathbf{c}_j(\mathbf{r}), \qquad i = x, y, \quad j = 1, 2 \\
&\langle\mathbf{c}_j\rangle^f = 0
\end{aligned} \tag{18}$$

Where $\theta_1 = -1$ and $\theta_2 = 1$. This leads to the following problem for φ

$$\begin{aligned}
\nabla\cdot(K\nabla\varphi) &= 0, \quad \text{in} \quad S_\mathrm{f} \\
\varphi(\mathbf{r}+l_i) &= \varphi(\mathbf{r}), \quad i = x, y \\
\langle\varphi\rangle^f &= 0.
\end{aligned}$$

from which it is easy to show that $\varphi = 0$, (Whitaker, 1999). The closure problems can be simplified further by using the periodicity of the gradient of the average of h, $\nabla\langle h\rangle = 0$ at the scale of the averaging volume. Then using the averaging theorem for ∇h

$$\langle\nabla h\rangle = \frac{1}{S}\int_{A_\mathrm{sf}} h\mathbf{n}_\mathrm{sf}\,\mathrm{d}A = \langle\nabla h_2\rangle - \langle\nabla h_1\rangle = 0.$$

Because, by definition $h = 0$ on $A_{\rm sf}$. Then using $\langle \nabla h_2 \rangle = \langle \nabla h_1 \rangle$ and $\widetilde{h} = \widetilde{h}_2 - \widetilde{h}_1$ it is easy to see that problem 2 and problem 3 (18) can be expressed anti-symmetrically

$$\frac{1}{12}\nabla\cdot\left(K\nabla \mathbf{c}_j - \langle K\nabla \mathbf{c}_j\rangle\right) = \theta_j\left(\langle \nabla h_+\rangle - \nabla h_+\right), \quad \text{in} \quad S_{\rm f}, \quad j = 1, 2 \tag{19}$$

Where $\theta_1 = -1$, $\theta_2 = 1$ and $h_+(\mathbf{x}) = (h_1 + h_2)/2$. From (19) it is now clear that problem 2 and problem 3 are left unchanged by the substitution $\mathbf{c}_1 = -\mathbf{c}_2$. Then, their solution fulfills $\mathbf{c}_1 = -\mathbf{c}_2$. We define the solution $\mathbf{c} = \mathbf{c}_1/6 = -\mathbf{c}_2/6$ and only need to solve a closure problem for $\mathbf{c}$. Moreover a further simplification can be found from the periodicity of ∇h_+, $\nabla\langle h_+\rangle = 0$ using the averaging theorem

$$\langle \nabla h_+\rangle = \frac{1}{S}\int_{A_{\rm sf}} h_+ \mathbf{n}_{\rm sf}\,\mathrm{d}A + \mathrm{O}(\epsilon^2) = \frac{h_+}{S}\int_{A_{\rm sf}} \mathbf{n}_{\rm sf}\,\mathrm{d}A + \mathrm{O}(\epsilon^2) = \mathrm{O}(\epsilon^2) \tag{20}$$

because, $A_{\rm sf} \subset \Omega$ then by definition on $A_{\rm sf}$, h_+ is constant (zero). As mentioned previously we are using the volume averaging method on the Riemannian surface Ω. Nevertheless, we have shown that a small slope ϵ of this surface provides a simple Euclidean version of the averaging theorem up to ϵ^2 terms. It would have been difficult to quantify such approximation from the direct formulation of the volume averaging theorem on the Euclidean Oxy plane for which

$$\langle \nabla h_+\rangle = \int_{\mathcal{A}_{\rm sf}} h_+ \mathbf{n}_{\rm sf}\,\mathrm{d}\mathcal{A} \neq 0.$$

From Equation (20) and because $\langle K\nabla \mathbf{b}\rangle$ and $\langle K\nabla \mathbf{c}\rangle$ can be treated as constants, one needs to solve the two following simplified closure problems

$$\begin{aligned}
&\nabla\cdot\left(\widetilde{K}\mathbf{I} + K\nabla \mathbf{b}\right) = 0, \quad \text{in} \quad S_{\rm f}\\
&\mathbf{b}(\mathbf{r} + l_i) = \mathbf{b}(\mathbf{r}), \quad i = x, y\\
&\langle \mathbf{b}\rangle^f = 0.
\end{aligned} \tag{21}$$

and

$$\begin{aligned}
&\nabla\cdot(K\nabla \mathbf{c}) = 2\nabla h_+, \quad \text{in} \quad S_{\rm f},\\
&\mathbf{c}(\mathbf{r} + l_i) = \mathbf{c}(\mathbf{r}), \quad i = x, y\\
&\langle \mathbf{c}\rangle^f = 0.
\end{aligned} \tag{22}$$

in which $\mathbf{c} = \mathbf{c_1}/6 = -\mathbf{c}_2/6$ and $h_+ = (h_1 + h_2)/2$. It is interesting to note that solving (21) only is sufficiant to obtain the macroscopisation of a pressure driven flow through a fracture (14). As a matter of facts, the next section will show how the macroscopic permeability tensor is related to the closure field $\mathbf{b}$, while the Couette macroscopic flow is related to $\mathbf{c}$.

2.2. AVERAGE FLOW MODEL

2.2.1. *Spatially Averaged Flow Model*

The closed form of Equation (14) is obtained by substituting the closure relation (16) for the averaged flux Equation (11). This yields

$$\langle \mathbf{q} \rangle = -\frac{1}{12\mu} \mathbf{K}^{*}.\nabla \langle p \rangle^{f} + \mathbf{C}.\left(\frac{\mathbf{U}_2 - \mathbf{U}_1}{2} \right) + \frac{\mathbf{U}_2 + \mathbf{U}_1}{2} \langle h \rangle, \tag{23}$$

in which,

$$\mathbf{K}^{*} = \langle K \rangle \mathbf{I} + \langle K \nabla \mathbf{b} \rangle, \qquad \mathbf{C} = \langle K \nabla \mathbf{c} \rangle. \tag{24}$$

At this stage it is worth noting that, as previously observed in Quintard and Whitaker (1987), the effective permeability tensor of a fracture is easily deduced from the closure field **b** and is a symmetric tensor. Because fracture walls, are generaly static, the temporal averaging is not necessary in this context, and relation (24) gives the effective permeability tensor, from the closure problem (21) which has to be solved.

Moreover, combining Equation (23) with Equation (8) leads to the closed form of the average Reynolds equation,

$$\begin{aligned} \nabla \cdot \left(\frac{1}{12\mu} \mathbf{K}^{*}.\nabla \langle p \rangle^{f} \right) \; &= \; (\nabla \cdot \mathbf{C}) \,.\left(\frac{\mathbf{U}_2 - \mathbf{U}_1}{2} \right) + \\ &+ \; \left(\frac{\mathbf{U}_2 + \mathbf{U}_1}{2} \right).\nabla \langle h \rangle - \mathbf{U}_2 \cdot \langle \nabla h_2 \rangle + \mathbf{U}_1 \cdot \langle \nabla h_1 \rangle \end{aligned} \tag{25}$$

It is worth noting that the obtained macroscopic Reynolds equation decouples the kinematic Couette contribution of the macroscopic velocities $\mathbf{U}_1$ and $\mathbf{U}_2$ from the microscopic contributions embodied in the closure fields **C** which is independant of the specified macroscopic kinematic conditions. Moreover, the microscopic contribution display an ‘intrinsic’ form, involving the surface velocity difference $\mathbf{U}_2 - \mathbf{U}_1$ in a relative cinematic frame, while the Couette contribution coming from the macroscopic spatial variations of the aperture field $\nabla \langle h \rangle$ involves a kinematic contribution in the laboratory frame proportional to $\mathbf{U}_2 + \mathbf{U}_1$.

2.2.2. *Spatially Averaged Shear*

After the flux and the pressure we now wish to form the averaged shear equation

$$\langle \tau \rangle = \left\langle \pm \frac{h}{2} \nabla p + \mu \frac{\mathbf{U}_2 - \mathbf{U}_1}{h} \right\rangle, \tag{26}$$

where the $+$ sign is associated with the shear stress at surface 2 and $-$ sign for surface 1. $\langle \tau \rangle$ stands for the average shear stress in the domain due to the fluid flow.

$$\langle \tau \rangle = \pm \left\langle \frac{h}{2} \nabla p \right\rangle + \mu \left\langle \frac{1}{h} \right\rangle (\mathbf{U}_2 - \mathbf{U}_1)\,. \tag{27}$$

By using the deviation decomposition $p = \langle p\rangle^f + \widetilde{p}$ and the closure form (16) we get to the expression

$$\langle\tau\rangle = \mu\left[\left\langle\frac{1}{h}\right\rangle\mathbf{I} \pm 3\langle -h\nabla\mathbf{c}\rangle\right].(\mathbf{U}_2 - \mathbf{U}_1) \pm \frac{1}{2}\left[\langle h\rangle\mathbf{I} + \langle h\nabla\mathbf{b}\rangle\right].\nabla\langle p\rangle^f. \quad (28)$$

The shear macroscopic stress vector displays a Couette and a Poiseuille contribution, for which pressure and kinematic conditions imposed at the macroscopic level are again decoupled from the microscopic contributions. As expected, the Couette shear stress displays an intrinsic form, involving the surface velocity difference $\mathbf{U}_2 - \mathbf{U}_1$, while the Poiseuille contribution involves an effective conductivity very similar to the effective permeability obtained in (24), using a permeability K proportional to the local aperture h.

2.3. TIME AVERAGE

The spatially averaged Equations (23), (25) and (28) generally depend on time since h, K, $\mathbf{c}$, $\mathbf{b}$, $\mathbf{K}^*$, $\mathbf{C}$, h_1, h_2 are all time dependent (one obvious exception is the case where one surface is a moving plane, while the other rough surface remains fixed). In principle, it is therefore necessary to perform a time average to obtain the average behaviors. In a companion paper (Letalleur *et al.*, 2000) we consider an example in which such a time average is necessary. As discussed in Letalleur *et al.* (2000), the time average is in fact needed when the two surfaces are strongly correlated to one another. In such a case, we define the time average as

$$\{\psi\} = \frac{1}{T}\int_T \psi \, \mathrm{d}t,$$

where T is the time over which the above equations are to be time averaged ($T \approx 2r_0/\left|U_2 - U_1\right|$). The time averaged of the above equation is straightforward. It yields

$$\{\langle\mathbf{q}\rangle\} = -\frac{1}{12\mu}\left\{\mathbf{K}^*\right\}.\nabla\langle p\rangle^f + \{\mathbf{C}\}.\left(\frac{\mathbf{U}_2 - \mathbf{U}_1}{2}\right) + \frac{\mathbf{U}_2 + \mathbf{U}_1}{2}\{\langle h\rangle\}, \quad (29)$$

$$\begin{aligned}\nabla\cdot\left(\frac{1}{12\mu}\{\mathbf{K}^*\}\nabla\langle p\rangle^f\right) &= \{(\nabla\cdot\mathbf{C})\}\cdot\left(\frac{\mathbf{U}_2 - \mathbf{U}_1}{2}\right) + \left(\frac{\mathbf{U}_2 + \mathbf{U}_1}{2}\right)\cdot\{\nabla\langle h\rangle\} \\ &\quad -\mathbf{U}_2\cdot\{\langle\nabla h_2\rangle\} + \mathbf{U}_1\cdot\{\langle\nabla h_1\rangle\}, \end{aligned} \quad (30)$$

$$\begin{aligned}\{\langle\tau\rangle\} &= \mu\left[\left\{\left\langle\frac{1}{h}\right\rangle\right\}\mathbf{I} \pm 3\{\langle -h\nabla\mathbf{c}\rangle\}\right]\cdot(\mathbf{U}_2 - \mathbf{U}_1) \pm \frac{1}{2}[\{\langle h\rangle\}\mathbf{I} + \\ &\quad +\{\langle h\nabla\mathbf{b}\rangle\}]\cdot\nabla\langle p\rangle^f. \end{aligned} \quad (31)$$

Here we have assumed that the characteristic times of variation of $\nabla\langle p\rangle^f$, $\mathbf{U}_2$, $\mathbf{U}_1$, if any, were much larger that T. It might be interesting to consider situations where these characteristic times are still significantly larger than t_d (for the Reynolds equation to be valid and the space average feasible before the time average) but of the order of, smaller than, T. This would introduce additional terms in the above time-space averaged equations.

3. Discussion

Time average has almost never been considered in previous works on average flow models based on the Reynolds equation. One exception is Elrod's work (1979), where the time averaging is clearly identified. As correctly noted by Elrod, the time average is necessary when the surface roughness and the surface motion are such that the time average cross covariance of the surfaces is not equal to zero. In all the previous works, that is Patir and Cheng (1979), Peeken *et al.* (1997) and references therein, the surfaces were not considered as intercorrelated. This explains why the time average was not considered. Time average is sometimes hidden in the ensemble average procedure, with an implicit ergodic assumption, see for instance Peeken *et al.* (1997). The need for a time averaging has already been noticed in a different context related to suspensions macroscopisation (Adler *et al.*, 1985). In this work, even if the pressure gradient is replaced by the external forces applied on the particules, Adler and Brenner grasp that time averaging cannot be cast in a spatial average procedure but even more that a time averaging procedure could cure some theoretical singularity arrising from a single spatial averaging in the macroscopic stress. The influence of a spatio-temporal averaging procedure on flow factors for a very simple sinusoidal geometry carried out in Letalleur *et al.* (2000) shows similarly a clear impact on the singular stress behavior.

In order to compare with previous models, we assume, in the rest of the present paper, that the time average is not necessary. Under these circumstances, the average Equations, (23), (25) and (28), can be viewed as generalizations of the empirical average model proposed by Patir and Cheng. First, the directions of motion of the surfaces are not necessarily the same. Second, as pointed out in the introduction, the influence of roughness must be taken into account through tensors ($\mathbf{K}^*$, $\mathbf{C}$, ..., in our model) and not scalar coefficients, as in Patir and Cheng's model. This allows one to deal with situations where the influence of roughness results in an average flow which does not follow the direction of the average pressure gradient. It is also interesting to note that our model becomes identical to the one derived by Bayada and Chambat (1988) if one considers the special case where surface 2 is smooth and moving whereas surface 1 is rough and fixed in the case where contacts are forbidden.

Patir and Cheng's model has been very popular in the field of lubrication. In addition to proposing average equations, they have introduced the concept of 'flow factors' that has also become popular. The 'flow factors', or more correctly the 'flow tensors', are simply a dimensionless expression of the tensors that appear in the average equations. In order to compare our model with the one of Patir and Cheng, we consider the special case where the off-diagonal terms are negligible (i.e. isotropic systems or anisotropic systems for which the main directions of anisotropy are parallel to the coordinate axis) and $\mathbf{U}_2$ and $\mathbf{U}_1$ are parallel to the x axis. In this case, our average flow model takes this form

$$\begin{aligned}
\langle \mathbf{q}_x \rangle &= - \left(\frac{\mathbf{K}^*_{xx}}{12\mu} \frac{\partial \langle p \rangle^f}{\partial x} \right) + \mathbf{C}_{xx} \left(\frac{\mathbf{U}_{2x} - \mathbf{U}_{1x}}{2} \right) + \frac{\mathbf{U}_{2x} + \mathbf{U}_{1x}}{2} \langle h \rangle, \\
\langle \mathbf{q}_y \rangle &= - \left(\frac{\mathbf{K}^*_{yy}}{12\mu} \frac{\partial \langle p \rangle^f}{\partial y} \right),
\end{aligned} \tag{32}$$

$$\begin{aligned}
\frac{\partial}{\partial x} \left(\frac{\mathbf{K}^*_{xx}}{12\mu} \frac{\partial \langle p \rangle^f}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\mathbf{K}^*_{yy}}{12\mu} \frac{\partial \langle p \rangle^f}{\partial y} \right) &= \frac{\partial \mathbf{C}_{xx}}{\partial x} \left(\frac{\mathbf{U}_{2x} - \mathbf{U}_{1x}}{2} \right) + \\
+ \left(\frac{\mathbf{U}_{2x} + \mathbf{U}_{1x}}{2} \right) \frac{\partial \langle h \rangle}{\partial x} - \mathbf{U}_{2x} \left\langle \frac{\partial h_2}{\partial x} \right\rangle &+ \mathbf{U}_{1x} \left\langle \frac{\partial h_1}{\partial x} \right\rangle.
\end{aligned} \tag{33}$$

and the x component of the average shear stress reads

$$\langle \tau_x \rangle = \mu \left[\left\langle \frac{1}{h} \right\rangle \pm 3 \left\langle -h \frac{\partial \mathbf{c}_x}{\partial x} \right\rangle \right] (\mathbf{U}_{2x} - \mathbf{U}_{1x}) \pm \frac{1}{2} \left[\langle h \rangle + \left\langle h \frac{\partial \mathbf{b}_x}{\partial x} \right\rangle \right] \frac{\partial \langle p \rangle^f}{\partial x}. \tag{34}$$

In order to compare with Patir and Cheng model, we must note that $\overline{h}_T$, $\overline{q}_T$, $\overline{\tau}_T$ in Patir and Cheng notations correspond to $\langle h \rangle$, $\langle q \rangle$, $\langle \tau \rangle$ respectively. Comparing with Patir and Cheng model finally leads to express the flow factors as

$$\begin{aligned}
&\phi_x = \frac{\mathbf{K}^*_{xx}}{h_m^3}, \qquad \phi_y = \frac{\mathbf{K}^*_{yy}}{h_m^3}, \qquad \sigma \phi_s = -\mathbf{C}_{xx}\, \phi_f = h_m \left\langle \frac{1}{h} \right\rangle \\
&\phi_{fs} = 3 h_m \left\langle -h \frac{\partial \mathbf{c}_x}{\partial x} \right\rangle, \qquad \phi_{fp} = \frac{1}{h_m} \left[\langle h \rangle + \left\langle h \frac{\partial \mathbf{b}_x}{\partial x} \right\rangle \right].
\end{aligned} \tag{35}$$

where h_m is the distance between the mean planes of the two surfaces and σ is the composite rms roughness, which is related to the the standard deviations of each surface by

$$\sigma = \sqrt{\sigma_1^2 + \sigma_2^2}, \tag{36}$$

with $\sigma_i = \sqrt{(h_i - \langle h_i \rangle)^2}$ with $i = 1, 2$.

4. Flow Factors for Two-Dimensional Roughness

In general, flow factors are obtained from the numerical solution of closure problems. Here we consider the simple case of striated unidirectional roughness for which flow factors expression can be analytically derived. As illustrated in Figure 3, the strias are assumed to be parallel to the y axis while the sliding motion is the x direction. In this problem, the aperture h depends only on x.

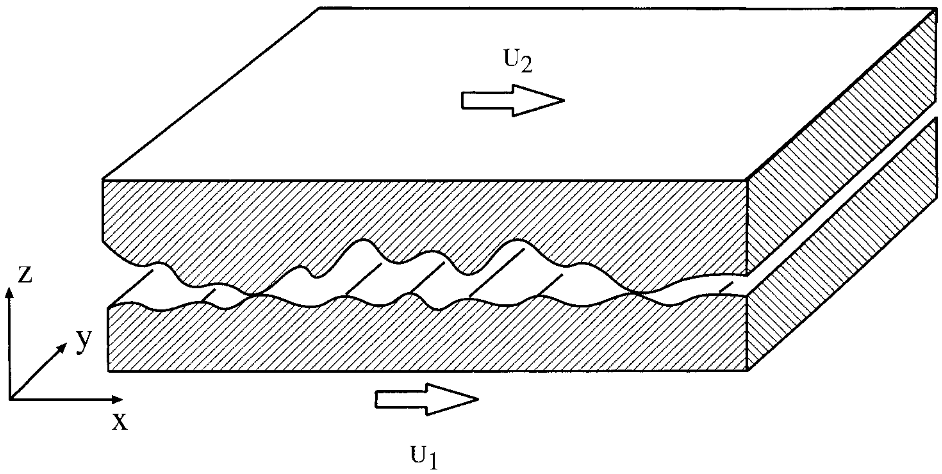

Figure 3. System of two striated surfaces in sliding motion. The striation are parallel to the y-axis.

The average flow model and the average shear read here,

$$\frac{\partial}{\partial x}\left(\phi_x \frac{h_m^3}{12\mu}\frac{\partial\langle p\rangle}{\partial x}\right) + \frac{\partial}{\partial y}\left(\phi_y \frac{h_m^3}{12\mu}\frac{\partial\langle p\rangle}{\partial y}\right) = \frac{U_{2x}+U_{1x}}{2}\frac{\partial\langle h\rangle}{\partial x} + \\ + \frac{U_{1x}-U_{2x}}{2}\sigma\frac{\partial\phi_{sx}}{\partial x},$$

$$\langle\tau_x\rangle = \mu\frac{U_{2x}-U_{1x}}{2}(\phi_{fx}\pm\phi_{fsx}) \pm \phi_{fpx}\frac{h_m}{2}\frac{\partial\langle p\rangle}{\partial x},$$
$$\langle\tau_y\rangle = \mu\frac{U_{2y}-U_{1y}}{2}(\phi_{fy}\pm\phi_{fsy}) \pm \phi_{fpy}\frac{h_m}{2}\frac{\partial\langle p\rangle}{\partial y},$$

Because h depends only on x, flow factors ϕ_y, ϕ_{fx}, ϕ_{fy} and ϕ_{fpy} are straightforwardly given by

$$\phi_y = \frac{\langle h^3\rangle}{h_m^3}, \qquad \phi_{fy} = \langle h^{-1}\rangle^f h_m, \qquad \phi_{fpy} = \frac{\langle h\rangle}{h_m} \tag{37}$$

The closure problems must be solved to determine the remaining flow factors. The closure problem is

$$\frac{\partial}{\partial x}\left(K\frac{\partial b_x}{\partial x}\right) = -\frac{\partial K}{\partial x}, \quad \text{in } S_\mathrm{f}$$
$$b_x(\mathbf{r}+l_i) = b_x(\mathbf{r}), \quad i = x, y \tag{38}$$

With $\langle b_x\rangle^f = 0$ and,

$$\frac{\partial}{\partial x}\left(K\frac{\partial c_x}{\partial x}\right) = \frac{\partial(h_1+h_2)}{\partial x}, \quad \text{in } S_\mathrm{f}$$
$$c_x(\mathbf{r}+l_i) = c_x(\mathbf{r}), \quad i = x, y \tag{39}$$

With $\langle c_x\rangle^f = 0$.

At this stage, it is useful to distinguish the no contact case from the case where contacts do exist.

4.1. NO CONTACT

Integration of Equation (38) gives

$$\frac{\partial b_x}{\partial x} = -1 + \frac{A}{K},$$

Constant A is obtained by taking the average, of this equation to get

$$-1 + A\left\langle \frac{1}{K} \right\rangle = 0,$$

This leads to express $\langle K \partial_x b_x \rangle$ as

$$\left\langle K \frac{\partial b_x}{\partial x} \right\rangle = -\langle K \rangle + \frac{1}{\left\langle \frac{1}{K} \right\rangle},$$

and $\langle h \partial_x b_x \rangle$ as

$$\left\langle h \frac{\partial b_x}{\partial x} \right\rangle = -\langle h \rangle + \frac{\left\langle \frac{1}{h^2} \right\rangle}{\left\langle \frac{1}{K} \right\rangle},$$

Integration of Equation (39) gives

$$\frac{\partial c_x}{\partial x} = \frac{(h_1 + h_2)}{K} + \frac{B}{K},$$

while taking the average of this equation leads to express the constant B as

$$B = \frac{-\left\langle \frac{h_1+h_2}{K} \right\rangle}{\left\langle \frac{1}{K} \right\rangle},$$

which leads to

$$\left\langle K \frac{\partial c_x}{\partial x} \right\rangle = \left[\langle h_1 + h_2 \rangle - \frac{\left\langle \frac{h_1+h_2}{K} \right\rangle}{\left\langle \frac{1}{K} \right\rangle} \right],$$

Using spatial decomposition $h_i = \langle h_i \rangle + \widetilde{h}_i$ with $i = 1, 2$ finally leads to express $\left\langle K \frac{\partial c_x}{\partial x} \right\rangle$ as

$$\left\langle K \frac{\partial c_x}{\partial x} \right\rangle = -\frac{\left\langle \frac{\widetilde{h}_1+\widetilde{h}_2}{K} \right\rangle}{\left\langle \frac{1}{K} \right\rangle},$$

and $\langle h \partial_x c_x \rangle$ as

$$\left\langle h \frac{\partial c_x}{\partial x} \right\rangle = \left[\left\langle \frac{\widetilde{h}_1 + \widetilde{h}_2}{h^2} \right\rangle - \left\langle \frac{\widetilde{h}_1 + \widetilde{h}_2}{K} \right\rangle \frac{\left\langle \frac{1}{h^2} \right\rangle}{\left\langle \frac{1}{K} \right\rangle} \right],$$

Flow factors are finally given by:

$$\begin{aligned}
\phi_x &= \frac{1}{h_m^3\langle h^{-3}\rangle},\\
\phi_y &= \frac{\langle h^3\rangle}{h_m^3}, \qquad \phi_{sx} = \frac{1}{\sigma\langle h^{-3}\rangle}\left\langle\frac{\widetilde{h}_1+\widetilde{h}_2}{h^3}\right\rangle, \qquad \phi_{fx} = \langle h^{-1}\rangle h_m,\\
\phi_{fpx} &= \frac{\langle h^{-2}\rangle}{h_m\langle h^{-3}\rangle},\\
\phi_{fsx} &= 3h_m\left[\left\langle\frac{\widetilde{h}_1+\widetilde{h}_2}{h^3}\right\rangle\frac{\langle h^{-2}\rangle}{\langle h^{-3}\rangle} - \left\langle\frac{\widetilde{h}_1+\widetilde{h}_2}{h^2}\right\rangle\right].
\end{aligned} \tag{40}$$

which is in complete agreement with the previously proposed expressions in the literature (see for example Elrod (1979), Tripp (1983), Bayada and Chambat (1988), Bayada and Faure (1989) among others).

4.2. WITH CONTACTS

Here, the closure problem for b_x is not to be considered since a macroscopic pressure gradient along x cannot exist in the presence of a contact line. We are only interested in the closure problem for c_x. Integration of Equation (39) gives

$$K\frac{\partial c_x}{\partial x} = (h_1+h_2) + B.$$

As $h=0$ at A_{sf}, the constant B can be expressed as

$$B = -(h_1+h_2) = -12h_0, \quad \text{on} \quad A_{sf}$$

in which h_0 is the average position of the contact lines in V_{f}. This leads to

$$\mathbf{C}_{xx} = \left\langle K\frac{\partial c_x}{\partial x}\right\rangle = [\langle h_1+h_2\rangle - 2h_0],$$

and

$$\left\langle h\frac{\partial c_x}{\partial x}\right\rangle = \left[\left\langle\frac{h_1+h_2}{h^2}\right\rangle - 12h_0\left\langle\frac{1}{h^2}\right\rangle\right],$$

which leads to express the flow factors as

$$\begin{aligned}
\phi_{sx} &= \frac{[2h_0 - \langle h_1+h_2\rangle]}{\sigma} \qquad \phi_{\mathrm{f}x} = \langle h^{-1}\rangle h_m\\
\phi_{fsx} &= 3h_m\left[2h_0\left\langle\frac{1}{h^2}\right\rangle - \left\langle\frac{h_1+h_2}{h^2}\right\rangle\right].
\end{aligned} \tag{41}$$

In the presence of contact, it should be noted that the integrals of the form $\langle h^{-n}\rangle$ with $n \geqslant 1$ may diverge in the vicinity of the contact zones. In fact convergence or divergence can be obtained depending on the behavior of h (as a function of the space coordinates) near contact. Again, this point raises the question of the validity of the Reynolds equation mentioned in Section 1.2. In fact, it may be observed that even in the context of the Stokes formulation that a continuum mechanics description is questionable when the aperture becomes on the order of molecular lengths. One reasonable solution is to consider that the Reynolds equation may be used to describe the flow and the shear due to the flow up to a lower bound in terms of aperture. The friction in contact zones is to be described by solid mechanics concept (including the possible presence of fluid molecules) while new models are needed to describe the flow and shear in regions where the aperture is between the aforementioned lower bound and zero. While this is a troublesome problem when there is friction between surfaces, it may be observed that the average flow model can be used with some confidence for determining the flow rate induced by the surface motion since the contribution of the regions when the latter are very close to contact is necessarily very small.

5. Conclusions

An average model for flow between rough surfaces in sliding motion has been derived by means of the method of volume averaging. Solid contact zones receive specific treatment. The analysis indicates that, under certain circumstances, the spatial average should be completed with a time average so that the average behaviors may be obtained. As illustrated in Letalleur *et al.* (2000), this solely depends on statistical inter-correlation of surfaces height. One interesting feature of the present work is to propose an effective formulation so that to compute transport coefficients by solving two closure problems defined over a representative region of the aperture field. The representative region is viewed as the unit cell of a spatially periodic system. This leads to impose periodicity boundary conditions. From the numerous works performed within the framework of the volume averaging method, see Whitaker (1999), and references therein, it is known that the influence of those boundary conditions on the effective properties is very weak. In this way, the cumbersome questions about the boundary conditions to be imposed for the numerical determination of flow factors, see for instance Teale and Lebeck (1980), Lunde and Tonder (1997), are avoided and moreover enlightened.

Our average flow model rests upon the assumption that the Reynolds equation is valid at the roughness scale. Although this assumption has been considered without discussion in many previous works dealing with rough surfaces in sliding motion with contacts, its validity would deserve to be explored in more detail. Moreover, surfaces deformation is essentially ignored in the present work. This is a serious limitation. In fact, taking deformation into account could affect not only the flow factors determination, see Knoll *et al.* (1998), but eventually the average equations form. It may be surmised that the time dependence could not be treated as trivially as here when dealing with time dependent local deformation.

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Appendix A

Let us consider the surface Ω defined on a Cartesian parametrisation that we will note here $Ox^1x^2x^3$ such that $x^3 = h_+(x^1, x^2)$ define Ω. Any point $\mathbf{u}$ on surface Ω can be parametrized simply $\mathbf{u} = (x^1, x^2, h_+(x^1, x^2))$. Hence two tangential vectors can be defined at any point $\mathbf{u}$ with

$$\mathbf{u}_i = \frac{\partial \mathbf{u}}{\partial x^i}, i = 1, 2$$

Explicitly $\mathbf{u}_1 = (1, 0, \partial_{x^1} h_+(x^1, x^2))$, and $\mathbf{u}_2 = (0, 1, \partial_{x^2} h_+(x^1, x^2))$. One can easily compute the metric tensor g_{ij} associated with Ω (see for example Frankel (1997)

$$g_{ij} = \mathbf{u}_i \cdot \mathbf{u}_j = \Sigma_{\alpha=1}^{3} \frac{\partial u^\alpha}{\partial x^i} \frac{\partial u^\alpha}{\partial x^j},$$

that is

$$g_{ij} \equiv \begin{pmatrix} 1 + (\partial_{x^1} h_+)^2 & \partial_{x^1} h_+ \partial_{x^2} h_+ \\ \partial_{x^1} h_+ \partial_{x^2} h_+ & 1 + (\partial_{x^2} h_+)^2 \end{pmatrix}.$$

Now, the surface Ω, has small slopes, that is slow variations in the direction x^1 and x^2. Thus we can write that $x^3 = h_+(X^1, X^2) = h_+(\epsilon x^1, \epsilon x^2)$ and then expand the metric in power of ϵ

$$g_{ij} = \delta_{ij} + \epsilon^2 g_{ij}^{(2)},$$

with δ_{ij} the identity tensor and

$$g_{ij}^{(2)} \equiv \begin{pmatrix} (\partial_{x^1} h_+)^2 & \partial_{x^1} h_+ \partial_{x^2} h_+ \\ \partial_{x^1} h_+ \partial_{x^2} h_+ & (\partial_{x^2} h_+)^2 \end{pmatrix}.$$

The generalization of differentiation on non-Euclidean spaces is the covariant derivative. Such differential operator can be computed using Christofell symbols $\Gamma^i_{j\lambda}$. For any covariant vector $\mathbf{a}$, with component a_i, the covariant derivative along the λ component is given by

$$\mathcal{D}_\lambda a_i = \partial_{\lambda a_i} + \Gamma^i_{j\lambda} a_j,$$

Where $\Gamma^i_{j\lambda} = 1/2 g^{i\sigma}(\partial_\lambda g^{j\sigma} + \partial_j g^{\sigma\lambda} - \partial_\sigma g^{\lambda j})$ using the repeated index convention. Christoffel symbols can similarly be expanded in ϵ,

$$\Gamma^i_{j\lambda} = \epsilon^2 \Gamma^{i(2)}_{j\lambda} + \epsilon^4 \Gamma^{i(4)}_{j\lambda},$$

where, for example

$$\begin{aligned}\Gamma^{i(2)}_{j\lambda} &= \frac{1}{2}\delta^{i\sigma}(\partial_\lambda g^{j\sigma(2)} + \partial_j g^{\sigma\lambda(1)} - \partial_\sigma g^{\lambda j(2)}) \\ &= \frac{1}{2}(\partial_\lambda g^{ji(2)} + \partial_j g^{i\lambda(2)} - \partial_i g^{\lambda j(2)}),\end{aligned}$$

Then it is easy to deduce that, from our ϵ expansion

$$\mathcal{D}_\lambda a_i = \partial_\lambda a_i + \mathrm{O}(\epsilon^2).$$

the covariant derivative and the Euclidean one differs by an ϵ^2 correction. This is obviously the same for integral operators. Then, integro-differential operators can be applied on Ω with their standard Euclidean formulation up to ϵ^2 corrections.

Appendix B

In this appendix, it is shown that

$$\langle K\nabla p\rangle = \langle K\rangle\nabla\langle p\rangle^f + \langle K\nabla\widetilde{p}\rangle, \tag{42}$$

that is $\left\langle K\nabla\langle p\rangle^f\right\rangle = \langle K\rangle\nabla\langle p\rangle^f$

The procedure (Whitaker, 1999) consists in using the following Taylor series expansion about the centroid of the averaging volume,

$$\nabla\langle p\rangle^f = \nabla\langle p\rangle^f|_x + \mathbf{y}_f.\nabla\nabla\langle p\rangle^f|_x + \frac{1}{2}\mathbf{y}_f\mathbf{y}_f : \nabla\nabla\nabla\langle p\rangle^f|_x \ldots, \tag{43}$$

in which $\mathbf{y_f}$ is a relative position vector locating a point in the fluid-phase relative to the centroid $\mathbf{x}$ of the averaging volume.

Substitution of this result into $\left\langle K\nabla\langle p\rangle^f\right\rangle$ leads to

$$\begin{aligned}\left\langle K\nabla\langle p\rangle^f\right\rangle &= \langle K\rangle\nabla\langle p\rangle^f|_x + \\ &\quad +\langle K\mathbf{y}_f\rangle.\nabla\nabla\langle p\rangle^f|_x + \frac{1}{2}\langle K\,\mathbf{y}_f\mathbf{y}_f\rangle : \nabla\nabla\nabla\langle p\rangle^f|_x.\end{aligned} \tag{44}$$

The next step is based on the following order of magnitude estimates

$$\nabla\nabla\langle p\rangle^f|_x = \mathrm{O}\left[\frac{\Delta(\nabla\langle p\rangle^f)}{L_p}\right],$$

$$\nabla\nabla\nabla\langle p\rangle^f|_x = \mathrm{O}\left[\frac{\Delta(\nabla\langle p\rangle^f)}{L_p^2}\right],$$

in which L_p represents a characteristic length associated with $\nabla\langle p\rangle^f$. The spatial moments $\langle \mathbf{y}\rangle$, $\langle \mathbf{yy}\rangle, \ldots$ have been studied by Quintard and Whitaker (1994). On the basis of their study, one finally obtains the following order of magnitude estimates

$$\langle K\mathbf{y}_f\rangle.\nabla\nabla\langle p\rangle^f|_x = \mathrm{O}\left[\langle K\rangle \frac{r_0}{L_p}\nabla\langle p\rangle^f\right], \tag{45}$$

$$\frac{1}{2}\langle K\mathbf{y}_f\mathbf{y}_f\rangle : \nabla\nabla\nabla\langle p\rangle^f|_x = \mathrm{O}\left[\langle K\rangle \left(\frac{r_0}{L_p}\right)^2 \nabla\langle p\rangle^f\right], \tag{46}$$

and similar estimates for higher order terms. These estimates show that the other terms in the r.h.s. of Equation (44) are negligible compared to the leading term provide that the following length-scale constraint is satisfied

$$\frac{r_0}{L_p} \ll 1.$$

We now show that $\nabla\langle p\rangle^f$ can also be considered as a constant over the averaging volume provide that $r_0/L_p \ll 1$.

The developments are classical in the context of the volume averaging method (Whitaker, 1999). One starts from Equation (43)

$$\nabla\langle p\rangle^f = \nabla\langle p\rangle^f|_x + \mathbf{y}_f.\nabla\nabla\langle p\rangle^f|_x + \frac{1}{2}\mathbf{y}_f\mathbf{y}_f : \nabla\nabla\nabla\langle p\rangle^f|_x \ldots . \tag{47}$$

The following estimates

$$\mathbf{y}_f.\nabla\nabla\langle p\rangle^f|_x = \mathrm{O}\left[\frac{r_0}{L_p}\nabla\langle p\rangle^f\right] \tag{48}$$

$$\frac{1}{2}\mathbf{y}_f\mathbf{y}_f : \nabla\nabla\nabla\langle p\rangle^f|_x = \mathrm{O}\left[\left(\frac{r_0}{L_p}\right)^2 \nabla\langle p\rangle^f\right] \tag{49}$$

show that under the length-scale constraint $r_0/L_p \ll 1$, the source term $\nabla\langle p\rangle^f$ can be considered as a constant over S.

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